Theoretical and computational bounds for m-cycles of the $3n+1$ problem

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Abstract

An m-cycle of the $3n+1$-problem is defined as a periodic orbit with m local minima. In this article we derive lower and upper bounds for the cycle length and the elements of (hypothetical) m-cycles. In particular, we prove that there do not exist nontrivial m-cycles for $1 \leq m \leq 75$. Our proofs are based on transcendental number theory, computational diophantine approximation techniques, and a not straightforward generalization of the approach of Steiner and Simons on 1-cycles and 2-cycles respectively.

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1 Introduction

1.1 General

The $3n+1$-problem is a well known problem in elementary number theory. Let n be a natural number, and consider the sequence generated by iteration of $T(n) = \frac{1}{2}(3n+1)$ if n is odd, $T(n) = \frac{1}{2}n$ is n is even. Numerical verification indicates that for all natural numbers finally the cycle $\{1, 2\}$ appears. A proof of this so called $3n+1$-conjecture is lacking sofar, in spite of various approaches to the problem. See Lagarias [La] and Wirsching [Wi] for extensive overviews on the $3n+1$-problem.

We call a cycle an m-cycle if the numbers in it appear in m sequences, each consisting of an increasing subsequence of odd numbers, followed by a decreasing subsequence of even numbers.

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Let such an m-cycle contain in total K odd numbers and L even numbers. The cycle $\{1,2\}$ is a 1-cycle, and in 1977 Steiner [St] proved that other 1-cycles do not exist. Any m-cycle containing natural numbers greater than 2 is called nontrivial, and in 2004 Simons [Si] proved the nonexistence of nontrivial 2-cycles.

In this paper we generalize the approach of Steiner and Simons, and for an arbitrary value of m we derive the following:

1. An upper bound for $\Lambda = (K+L)\log 2 - K\log 3$ that is exponential in K, following from estimates for m-cycles;
2. A lower bound for Λ that is subexponential in K, following from transcendence theory;
3. Upper bounds for K and L, and for the minimal element $x_{\min}$ of the cycle, following from comparing the upper and lower bounds for Λ. These upper bounds appear to be exponential in m.

Further, for 'small' and 'medium' values of m we derive

4. Lower bounds for K and L, following from brute force computations and diophantine approximation techniques (continued fractions, lattice basis reduction) applied to $\dfrac{\log 3}{\log 2}$.

For 'small' values of m (up to $m \le 75$) the lower bound of 4. is larger than the upper bound of 3. For 'medium' values of m (up to $m \le 515\,620$) the diophantine approximation techniques lead to an improvement of the upper bounds for K, L and $x_{\min}$, which remain exponential in m.

1.2 m-Cycles of the $3n+1$-problem

We study periodic sequences $\{T^k(n)\}$, i.e. we do not consider unbounded sequences. Without loss of generality we're not interested in nonperiodic parts, so if necessary we replace n by some $T^k(n)$ which is in the periodic part. Hence we may assume that the sequence is purely periodic, i.e. there exists an integer $p \ge 1$ such that in the periodic sequence $\{n, T(n), T^2(n), \ldots, T^{p-1}(n), T^p(n), \ldots\}$ it is the case that $T^p(n) = n$ (we may take p minimal with this property, but for our arguments that is not essential). We consider only the period, i.e. $\{n, T(n), T^2(n), \ldots, T^{p-1}(n)\}$. We further may assume that $T^0(n) = n$ is at a local minimum in the sequence (not necessarily the global minimum).

Let there be m local minima in the periodic sequence, with indices $t_0, t_1, \ldots, t_{m-1}$ such that $0 = t_0 < t_1 < \ldots < t_{m-1} < p$. Then there are also m local maxima, say with indices $s_0, s_1, \ldots, s_{m-1}$. As each maximum lies in between two minima, we may assume $0 = t_0 < s_0 < t_1 < s_1 < \ldots < t_{m-1} < s_{m-1} \le p-1$. We call such a periodic sequence an *m-cycle*. Define x_i, y_i as the values of the local minima and maxima, viz.

$$x_i = T^{t_i}(n), \qquad y_i = T^{s_i}(n).$$

We put $k_i = s_i - t_i$ for $i = 0, \ldots, m-1$, and $\ell_i = t_{i+1} - s_i$ for $i = 0, \ldots, m-2$, and $\ell_{m-1} = p + t_0 - s_{m-1}$. Further we put

$$K = \sum_{i=0}^{m-1} k_i, \qquad L = \sum_{i=0}^{m-1} \ell_i.$$

The sequence thus starts with an odd number x_0, increases in k_0 steps until an even number y_0 is encountered, then decreases in ℓ_0 steps until an odd number x_1 is encountered, again increases in k_1 steps until an even number y_1 is encountered, etcetera.

We call the m-fold repeated 1-cycle $\{1,2,1,2,\ldots,1,2\}$ the trivial m-cycle.

1.3 Crandall's lemma

For a nontrivial m-cycle, put $x_{\min} = \min\{x_0, x_1, \ldots, x_{m-1}\}$. By X_0 we denote a lower bound for $x_{\min}$ that is known to be true at a certain point in time. At the moment of writing this (August 31, 2010) it is known from extensive computations by Tomás Oliveira e Silva [OS] that

$$x_{\min} > X_0 = 5 \cdot 2^{60} > 5.7646 \cdot 10^{18}.$$

Eric Roosendaal's independent computations [Ro] have reached $x_{\min} > 890 \times 2^{50} > 1.0020 \cdot 10^{18}$. The computations are going on[1]. The rate of improvement over the period June 2004 to January 2009 has been on average approximately $3.5 \cdot 10^{15}$ per day (see [OS]).

Let $\delta = \dfrac{\log 3}{\log 2} = 1.5849\ldots$. Throughout this paper this number δ plays a central role. Let its continued fraction expansion be given by $\delta = [a_0, a_1, a_2, \ldots] = [1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, \ldots]$, with convergents $\dfrac{p_n}{q_n} = [a_0, a_1, a_2, \ldots, a_n]$ for $n = 0, 1, \ldots$.

Crandall [Cr] showed the following result.

Lemma 1 (Crandall, 1978)
If $\dfrac{p_n}{q_n}$ is any convergent to δ with $n \geq 4$, then for a nontrivial m-cycle

$$K > \min\left\{q_n, \frac{2x_{min}}{q_n + q_{n+1}}\right\}.$$

As a consequence, we have the following result.

Corollary 2

$$K > 9.0240 \cdot 10^8.$$

Proof. This follows immediately from Lemma 1, the present value for X_0, and the fact that $q_{19} = 397\,573\,379$, $q_{20} = 6\,189\,245\,291$, and $q_{21} = 6\,586\,818\,670$. □

In this paper we generalize Crandall's Lemma for m-cycles, which in general results in sharper lower bounds for K.

1.4 Known results on the nonexistence of cycles

Steiner [St] proved in 1977 the nonexistence of nontrivial 1-cycles. He assumes the existence of a 1-cycle with k odd numbers and ℓ even numbers, and proves the following partial results:

1. An inequality for the ratio $\dfrac{k+\ell}{k}$;
2. A numerical lower bound for k, from which it follows that $\dfrac{k+\ell}{k}$ must be a convergent from the continued fraction expansion of δ;
3. An upper bound for k from a theorem of Baker [Ba] on linear forms in two logarithms;
4. A (very effective) lower bound for the partial quotients in the continued fraction expansion of δ.

[1] It seems that Oliveira da Silva has stopped since January 2009.

Numerical calculation of partial quotients then shows that the only 1-cycle that satisfies these conditions is the trivial one.

Crucial in Steiner's proof for the nonexistence of 1-cycles is (implicitly) the inequality $0 < (k+\ell)\log 2 - k\log 3 < \dfrac{1}{x_{\min}}$. The right hand side is exponentially small in k, since the existence of k successive odd numbers starting with x_0 implies that $x_0 = a2^k - 1 \geq 2^k - 1$. Hence $\dfrac{k+\ell}{k}$ must be a convergent of δ for $k \geq 5$. This inequality has a "natural" generalization for 2-cycles in the form of $0 < (K+L)\log 2 - K\log 3 < \dfrac{1}{x_0} + \dfrac{1}{x_1}$, however the convergent argument fails because k_0 or k_1 can be small even if K is large. As has been remarked by Lagarias [La], the result of Steiner's proof seems rather weak, considering the strength of the underlying number theory.

Simons [Si] proved in 2004 the nonexistence of nontrivial 2-cycles. By exploiting the average values of k_0 and k_1 he derived for the expression $\dfrac{1}{x_0} + \dfrac{1}{x_1}$ an effective upper bound of the form $ce^{-d(K+L)}$, where c and d are positive constants. He generalizes Steiner's approach and derives

1. A generalized inequality for the ratio $\dfrac{K+L}{K}$;
2. A numerical lower bound for K, from which it follows that $\dfrac{K+L}{K}$ must be a convergent from the continued fraction expansion of δ;
3. An upper bound for K by applying a theorem of Laurent, Mignotte and Nesterenko [LMN] on linear forms in two logarithms;
4. A lower bound for the partial quotients in the continued fraction expansion of δ.

Steiner's original numerical verification finally shows that the only 2-cycle that satisfies these conditions is the trivial 2-cycle $\{1, 2, 1, 2\}$.

This approach however fails to prove the nonexistence of m-cycles for $m > 2$, because then the coefficient d in Simons' upper bound becomes negative, which makes the upper bound ineffective.

Many partial results on the (non)existence of cycles for the $3n+1$-problem, as well as for generalizations, have been conjectured and proved by applying a scala of theoretical methods, see Lagarias [La] and Wirsching [Wi]. In particular, using transcendence methods like we do, Brox [Br] showed that there are only finitely many m-cycles with $m < 2\log K$, and from that he derived the result that for each m there are only finitely many m-cycles (see Theorem 3(a) below).

1.5 Lower and upper bounds on cycle elements and lengths

In this paper we extend the cited results of Steiner [St] and Simons [Si], and prove that for each m there are only finitely many m-cycles. Indeed, extending the result of Brox [Br], for m-cycles we derive explicit upper bounds, depending only on m, for the values of K, L and $x_{\min}$.

By doing extensive computations we also derive new lower bounds for these values. Then, combining upper and lower bounds, we prove that there exist no nontrivial m-cycles for $m \leq 75$. For $m = 76$ and $m = 77$ we give possible solutions, which will be excluded when exterior computations á la [OS] and [Ro] lead to new values for X_0. For $m \geq 78$ we derive explicit lower and upper bounds for the cycle length and for the numbers in the cycle.

Our main result is the following theorem.

Theorem 3 (Main Theorem)
For an m-cycle for the $3n+1$-problem, let K, L, x_{min} be defined as above.

(a) *(Brox [Br]) For any m there are only finitely many m-cycles.*

(b) *For $1 \le m \le 75$ there do not exist nontrivial m-cycles.*

(c) *For $76 \le m \le 77$ the only possible nontrivial m-cycles satisfy $x_{min} > 5.7646 \cdot 10^{18}$ and*

m	$K =$	$L =$	$x_{min} <$
76	117 972 833 293 231 014	69 009 683 580 368 485	$6.2044 \cdot 10^{18}$
	124 207 383 220 472 977	72 656 661 496 678 846	$1.0825 \cdot 10^{19}$
	130 441 933 147 714 940	76 303 639 412 989 207	$4.2381 \cdot 10^{19}$
77	117 972 833 293 231 014	69 009 683 580 368 485	$6.2860 \cdot 10^{18}$
	124 207 383 220 472 977	72 656 661 496 678 846	$1.0967 \cdot 10^{19}$
	130 441 933 147 714 940	76 303 639 412 989 207	$4.2939 \cdot 10^{19}$
	254 649 316 368 187 917	148 960 300 909 668 053	$8.7355 \cdot 10^{18}$

(d) *For $m \ge 78$ the possible nontrivial m-cycles satisfy*

if $78 \le m \le 90$ then

$$\begin{aligned}
1.1173 \cdot 10^{17} &< K < 1.3993\, m\delta^m &&< e^{0.46057m+\log m+0.33593},\\
6.1715 \cdot 10^{16} &< L < 0.81850\, m\delta^m &&< e^{0.46057m+\log m-0.20028},\\
5.7646 \cdot 10^{18} &< x_{min} < 339.14\, m^2\delta^m &&< e^{0.46057m+2\log m+5.8265},
\end{aligned}$$

if $91 \le m \le 515\,619$ then

$$\begin{aligned}
7.5311 \cdot 10^{11} &< K < 1.4784\, m\delta^m &&< e^{0.46057m+\log m+0.39095},\\
4.4054 \cdot 10^{11} &< L < 0.86480\, m\delta^m &&< e^{0.46057m+\log m-0.14525},\\
5.7646 \cdot 10^{18} &< x_{min} < 5.1825 \cdot 10^7\, m^2\delta^m &&< e^{0.46057m+2\log m+17.764},
\end{aligned}$$

if $515\,620 \le m \le 527\,875\,034$ then

$$\begin{aligned}
9.0240 \cdot 10^{8} &< K < 15.109\, m\delta^m &&< e^{0.46057m+\log m+2.7153},\\
5.2787 \cdot 10^{8} &< L < 8.8379\, m\delta^m &&< e^{0.46057m+\log m+2.1791},\\
5.7646 \cdot 10^{18} &< x_{min} < &&< e^{6.1260m},
\end{aligned}$$

if $m \ge 527\,875\,035$ then

$$\begin{aligned}
1.7095\, m &< K < 15.108\, m\delta^m &&< e^{0.46057m+\log m+2.7152},\\
m &\le L < 8.8372\, m\delta^m &&< e^{0.46057m+\log m+2.1790},\\
5.7646 \cdot 10^{18} &< x_{min} < &&< e^{6.1255m}.
\end{aligned}$$

In the cases $91 \le m \le 527\,875\,034$ the lower bounds can be a bit more fine-tuned (see Corollary 11 below). The upper bounds in terms of powers of e are provided to ease comparison.

We expect that, with current technology and efforts, $m = 76$ and $m = 77$ can be expected to be solved within 30 years, and that $m = 78$ and beyond may take considerably longer to be finished.

2 Conditions for the existence of an m-cycle

2.1 The chain equation

We first describe an m-cycle in more detail.

The existence of k_i successive odd numbers starting with x_i implies that $x_i \equiv -1 \pmod{2^{k_i}}$, i.e. $x_i = 2^{k_i} a_i - 1$ for some integer $a_i \ge 1$. Going up from a local minimum x_i to the next local maximum y_i is now expressed in the formula $y_i = 3^{k_i} a_i - 1$. Then going down to the next local

minimum x_{i+1} is done by $y_i = 2^{\ell_i} x_{i+1}$ (with $x_m = x_0$). Putting this together we arrive at the 'chain equation'

$$3^{k_i} a_i - 1 = 2^{k_{i+1}+\ell_i} a_{i+1} - 2^{\ell_i}$$

for $i = 0, 1, ..., m-1$. If we put $x_m = x_0$, i.e. $a_m = a_0$ and $k_m = k_0$, we impose the existence of an m-cycle.

Note that the chain equation with $m = 1$ is at the heart of Steiner's result [St]. From now on we take $m \geq 2$.

2.2 Integral and rational solutions

When we fix k_i, ℓ_i in the chain equation, we have m linear equations in the m variables a_i. We get the 'matrix equation'

$$M \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} 2^{\ell_0} - 1 \\ 2^{\ell_1} - 1 \\ \vdots \\ 2^{\ell_{m-1}} - 1 \end{pmatrix},$$

with the matrix M defined as

$$M = \begin{pmatrix} -3^{k_0} & 2^{k_1+\ell_0} & 0 & \cdots & 0 \\ 0 & -3^{k_1} & 2^{k_2+\ell_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & -3^{k_{m-2}} & 2^{k_{m-1}+\ell_{m-2}} \\ 2^{k_0+\ell_{m-1}} & 0 & \cdots & 0 & -3^{k_{m-1}} \end{pmatrix}.$$

Put

$$\Delta = 2^{K+L} - 3^K.$$

Then $\det(M) = (-1)^{m-1}\Delta$. Applying linear algebra, in fact using an argument of Böhm and Sontacchi [BS], we proceed to find the inverse matrix M^{-1}. Put $\Delta M^{-1} = (m_{i,j})_{i,j=0,1,...,m-1}$, $\alpha_{i,j} = \sum_{h=i+1}^{j^*} k_h$, $\beta_{i,j} = \sum_{h=i+1}^{j^*} \ell_{h-1}$, where we take $j^* = j$ if $i \leq j$, and $j^* = j + m$ if $i > j$, where the indices of k_h, ℓ_h are taken modulo m. Then we have $m_{i,j} = 2^{\alpha_{i,j}+\beta_{i,j}} 3^{K-k_i-\alpha_{i,j}}$, as is easily verified. In particular $m_{i,j} > 0$, and then the matrix equation and $a_i > 0$ imply that $\Delta > 0$.

It follows that m-cycles are in one-to-one correspondence with solutions k_i, ℓ_i, a_i of the matrix equation, with k_i, ℓ_i, a_i all positive integers. For a given combination of k_i and ℓ_i at most one such solution exists (there is exactly one solution with a_i rational[2], but in most cases the a_i are not integral). It follows that when we know upper bounds for K and L, there are only finitely many rational solutions left that can in principle be enumerated.

[2]Rational solutions with common denominator q correspond to solutions of the $3n+q$-problem.

3 Conditions on K and m from a linear form in logarithms

3.1 Introducing Λ

Like in Steiner's and Simons' proofs [St], [Si], the basis of our final result is a linear form in logarithms of integers, that for large m-cycles turns out to be too small to be possible. This linear form is

$$\Lambda = (K+L)\log 2 - K\log 3.$$

3.2 A first inequality for Λ

Lemma 4

$$0 < \Lambda < \sum_{i=1}^{m} \frac{1}{x_i}.$$

Proof. Rewrite the chain equation as

$$\frac{2^{k_{i+1}+\ell_i}}{3^{k_i}} \frac{a_{i+1}}{a_i} = 1 + \frac{2^{\ell_i}-1}{3^{k_i} a_i}.$$

Taking the product over all $i = 0, 1, \ldots, m-1$, and using the cyclicity, we get

$$\frac{2^{K+L}}{3^K} = \prod_{i=0}^{m-1} \left(1 + \frac{2^{\ell_i}-1}{3^{k_i} a_i}\right).$$

We now apply $\log(1+x) < x$ to each term in the product, and thus obtain

$$0 < \Lambda = \sum_{i=0}^{m-1} \log\left(1 + \frac{2^{\ell_i}-1}{3^{k_i} a_i}\right) < \sum_{i=0}^{m-1} \frac{2^{\ell_i}-1}{3^{k_i} a_i}.$$

The result now follows by $3^{k_i} a_i = y_i + 1 > y_i = 2^{\ell_i} x_{i+1} > (2^{\ell_i}-1)x_{i+1}$, and the cyclicity. □

We immediately have from Lemma 4

Corollary 5

$$0 < \Lambda < \frac{m}{x_{min}} \leq \frac{m}{X_0}.$$

Proof. Use Lemma 4 together with $x_i \geq x_{\min} \geq X_0$. □

3.3 Chaining

Now we will show that all x_i are exponentially large in terms of K (or, equivalently, $K+L$). To do so, we first show that all x_i are of about the same size, by 'chaining' them.

Put $b = \dfrac{1 + X_0^{-1}}{2^{1/\delta}}$. With the present value of X_0 we have $b = 0.64576\ldots$. In fact, the dependence on X_0 is negligible.

Lemma 6
For all $i = 0, 1, \ldots, m-1$ we have $x_{i+1} < b^\delta x_i^\delta$.

Proof. We have

$$\begin{aligned} x_{i+1} &= \frac{y_i}{2^{\ell_i}} < \frac{y_i+1}{2^{\ell_i}} = \frac{3^{k_i}a_i}{2^{\ell_i}} \le \frac{3^{k_i}}{2}a_i = \frac{3^{k_i}}{2}\frac{x_i+1}{2^{k_i}} = \left(\frac{3}{2}\right)^{k_i}\frac{x_i+1}{2} \\ &= \frac{1}{2}\left(2^{k_i}\right)^{\delta-1}(x_i+1). \end{aligned}$$

Now use $a_i \ge 1$ and $x_i \ge X_0$ to get

$$x_{i+1} < \frac{1}{2}\left(2^{k_i}a_i\right)^{\delta-1}(x_i+1) = \frac{1}{2}(x_i+1)^\delta \le \frac{1}{2}\left(1+X_0^{-1}\right)^\delta x_i^\delta = b^\delta x_i^\delta.$$

□

Note that the inequality of Lemma 6 holds cyclically. As a result we can estimate all x_i in terms of one x_i of our choice, say x_0.

3.4 Another inequality for Λ

Put

$$c_m = 2^{\frac{m}{\delta}\frac{\delta-1}{\delta^m-1}} b^{\frac{\delta}{\delta-1}-\frac{m}{\delta^m-1}}.$$

With the present value of X_0 we find that c_m decreases from $c_2 = 0.76479\ldots$ to $b^{\delta/(\delta-1)} = 0.30576\ldots$. When we also let X_0 tend to infinity, we get $c_m \to 2^{-1/(\delta-1)} = 0.30576\ldots$. Anyway, $c_m < 0.30577$ for $m \ge 30$.

As a consequence of Lemmas 4 and 6 we can now estimate Λ in terms of its coefficients, i.e. K.

Lemma 7

$$0 < \Lambda < m c_m 2^{-\frac{\delta-1}{\delta^m-1}K}.$$

Proof. Lemma 6 implies $x_i < b^{\delta+\delta^2+\ldots+\delta^i} x_0^{\delta^i}$ for $i = 1, 2, \ldots, m-1$. Hence

$$\prod_{i=0}^{m-1} x_i < b^{(m-1)\delta+(m-2)\delta^2+\ldots+\delta^{m-1}} x_0^{1+\delta+\delta^2+\ldots+\delta^{m-1}} = b^{\frac{\delta}{\delta-1}\left(\frac{\delta^m-1}{\delta-1}-m\right)} x_0^{\frac{\delta^m-1}{\delta-1}}.$$

On the other hand, also

$$\prod_{i=0}^{m-1} x_i = \prod_{i=0}^{m-1} \frac{x_i+1}{1+x_i^{-1}} \ge (1+X_0^{-1})^{-m} \prod_{i=0}^{m-1} 2^{k_i} a_i \ge (1+X_0^{-1})^{-m} 2^K,$$

where we simply estimated $a_i \ge 1$. Hence

$$x_0^{-\frac{\delta^m-1}{\delta-1}} < (1+X_0^{-1})^m b^{\frac{\delta}{\delta-1}\left(\frac{\delta^m-1}{\delta-1}-m\right)} 2^{-K} = c_m^{\frac{\delta^m-1}{\delta-1}} 2^{-K}.$$

Now we choose $x_0 = x_{\min}$, which we can do because of the cyclicity. Corollary 5 to Lemma 4 then shows that

$$0 < \Lambda < \frac{m}{x_{\min}} < m c_m 2^{-\frac{\delta-1}{\delta^m-1}K}.$$

□

Though strictly speaking the inequality of Lemma 7 depends on the value of X_0, this dependence is negligible.

4 Conditions on K and m from continued fractions

4.1 A useful lemma

A consequence of Corollary 5 to Lemma 4 is we have sharp lower and upper bounds for the ratios $\frac{K+L}{K}$, $\frac{K+L}{L}$ and $\frac{K}{L}$. This is useful not only in this section but also further in this paper.

Lemma 8

$$\begin{aligned} \delta K &< K+L &< 1.000001\,\delta K, \\ 0.999999\,\frac{\delta}{\delta-1}L &< K+L &< \frac{\delta}{\delta-1}L, \\ 0.999999\,\frac{1}{\delta-1}L &< K &< \frac{1}{\delta-1}L. \end{aligned}$$

Proof. By Corollary 5 and the present value of X_0 we have

$$0 < K + L - K\delta < \frac{m}{X_0 \log 2} \le \frac{K}{X_0 \log 2} < 10^{-18}K,$$

and the inequalities readily follow. □

4.2 Continued fraction results

Recall that we denote by $\frac{p_n}{q_n}$ the nth convergent to δ. Continued fraction theory shows that convergents are best approximations, i.e. any other approximation with smaller denominator is worse. Further necessary and sufficient inequalities for convergents are available. Indeed, we have the following results, the proofs of which can be found in many introductory texts on number theory (see e.g. [HW, Chapter 10]).

Lemma 9
(a) *If $\frac{p}{q}$ is a rational approximation to δ satisfying $|p - q\delta| < \frac{1}{2q}$, then $\frac{p}{q}$ is a convergent.*
(b) $|p_n - q_n\delta| > \frac{1}{q_n + q_{n+1}} > \frac{1}{(a_{n+1}+2)q_n}$.
(c) *If $\frac{p}{q}$ is a rational approximation to δ, and if $q \le q_n$, then $|p - q\delta| \ge |p_n - q_n\delta|$.*
(d) *If n is odd then $p_n - q_n\delta > 0$; if n is even then $p_n - q_n\delta < 0$.*

4.3 A generalization of Crandall's Lemma to m-cycles

With Corollary 5 we now can derive a result like Crandall's Lemma 1, which gives a lower bound for K that depends on m.

Lemma 10
If $q_n + q_{n+1} \le (\log 2)\dfrac{X_0}{m}$, *then* $K > q_n$.

Proof. Assume $K \le q_n$. By Lemma 9(c) and (b)

$$\Lambda = (\log 2)|(K+L) - K\delta| \ge (\log 2)|p_n - q_n\delta| > \frac{\log 2}{q_n + q_{n+1}} \ge \frac{m}{X_0},$$

which contradicts Corollary 5. □

Applying this with for each m the maximal n that satisfies the condition, leads to the following result.

Corollary 11

If	$m \le$	6	*then* $K > q_{34} > 2.6711 \times 10^{17}$.
If	$7 \le m \le$	10	*then* $K > q_{33} > 1.3044 \times 10^{17}$.
If	$11 \le m \le$	29	*then* $K > q_{32} > 6.2345 \times 10^{15}$.
If	$30 \le m \le$	333	*then* $K > q_{31} > 5.7509 \times 10^{15}$.
If	$334 \le m \le$	640	*then* $K > q_{30} > 4.8361 \times 10^{14}$.
If	$641 \le m \le$	4 367	*then* $K > q_{29} > 4.3116 \times 10^{14}$.
If	$4\,368 \le m \le$	8 262	*then* $K > q_{28} > 5.2449 \times 10^{13}$.
If	$8\,263 \le m \le$	62 412	*then* $K > q_{27} > 1.1571 \times 10^{13}$.
If	$62\,413 \le m \le$	225 312	*then* $K > q_{26} > 6.1624 \times 10^{12}$.
If	$225\,313 \le m \le$	345 300	*then* $K > q_{25} > 5.4093 \times 10^{12}$.
If	$345\,301 \le m \le$	648 401	*then* $K > q_{24} > 7.5311 \times 10^{11}$.
If	$648\,402 \le m \le$	4 486 354	*then* $K > q_{23} > 1.3752 \times 10^{11}$.
If	$4\,486\,355 \le m \le$	19 683 486	*then* $K > q_{22} > 6.5470 \times 10^{10}$.
If	$19\,683\,487 \le m \le$	55 451 899	*then* $K > q_{21} > 6.5868 \times 10^{9}$.
If	$55\,451\,900 \le m \le$	312 750 582	*then* $K > q_{20} > 6.1892 \times 10^{9}$.
If	$312\,750\,583 \le m \le$	527 875 034	*then* $K >$ 9.0240×10^{8}.
If	$527\,875\,035 \le m$		*then* $K > 1.7095m$.

Proof. All lines but the last two follow immediately from Lemma 10. The one but last line is Corollary 2 to Crandall's original Lemma 1. The last line follows by Lemma 8 from the trivial observation that $L \ge m$. □

Note that the bounds for m in Corollary 11 depend heavily on the value of X_0. Also note that Corollary 11 implies $K > 6.8580 \cdot 10^{16}\frac{1}{m}$.

5 Application of Transcendence Theory

Now that we know that Λ is exponentially small in terms of its coefficients, we can invoke deep but explicit results from transcendence theory, which tell us that linear forms in logarithms of integers cannot be too small in terms of their coefficients. Since Steiner published his paper [St] with essentially the same idea of applying transcendence theory to prove the nonexistence of nontrivial 1-cycles, transcendence theory has made substantial progress. For general linear forms $x\log a + y\log b$ with $x, y \in \mathbb{Z}$ and $a, b \in \mathbb{N}$ the best results today are the result of Laurent, Mignotte and Nesterenko [LMN] for small x, y, and for large x, y that of Matveev [Ma] (see also Nesterenko [Ne]). For our specific case $x\log 2 + y\log 3$ however, the result of Rhin [Rh] is best. From it we derive the following estimate.

Lemma 12

$$\Lambda > e^{-13.3(0.46057+\log K)}.$$

Proof. We apply the Proposition on p. 160 of [Rh] with $u_0 = 0$, $H = u_1 = K + L$, and $u_2 = -K$. Together with Lemma 8 the result follows. □

As a consequence of Lemma 12 we can also estimate the global minimum of an m-cycle in terms of K.

Corollary 13

$$x_{min} < m e^{13.3(0.46057+\log K)}.$$

Proof. Apply Corollary 5 and Lemma 12. □

6 Upper bounds for K, L and $x_{\min}$

6.1 Initial upper bound

Clearly Lemmas 7 and 12 are contradictory when K is large enough. In other words, they provide an upper bound for K, and then by Lemma 8 and Corollary 13 also for L and $x_{\min}$.

Lemma 14 *Let $x = K_1(m)$ be the largest solution of*

$$e^{-13.3(0.46057+\log x)} = m c_m 2^{-\frac{\delta-1}{\delta^m-1}x}.$$

Then

$$K < K_1(m).$$

Proof. Immediate from Lemmas 7 and 12 and the definition of $K_1(m)$. □

Let $k_1(m) = K_1(m)/(m\delta^m)$. Note that $k_1(m)$ is a decreasing function that for increasing m tends to the constant $\frac{13.3\log\delta}{\log 2(\delta-1)} = 15.107\ldots$ as $m \to \infty$. Indeed:

m	$k_1(m) <$	$K_1(m) <$	m	$k_1(m) <$	$K_1(m) <$
2	61.456	$3.0877 \cdot 10^{2}$	100 000	15.113	$1.1981 \cdot 10^{20\,008}$
10	35.843	$3.5858 \cdot 10^{4}$	515 619	15.109	$3.0475 \cdot 10^{103\,140}$
63	19.158	$4.8181 \cdot 10^{15}$	515 620	15.109	$4.8301 \cdot 10^{103\,140}$
64	19.102	$7.7350 \cdot 10^{15}$	1 000 000	15.108	$1.4816 \cdot 10^{200\,026}$
100	17.798	$1.7876 \cdot 10^{23}$	10 000 000	15.108	$1.2429 \cdot 10^{2\,000\,198}$
511	15.739	$1.3035 \cdot 10^{106}$	100 000 000	15.108	$2.1457 \cdot 10^{20\,001\,908}$
512	15.738	$2.0698 \cdot 10^{106}$	527 875 034	15.108	$7.1282 \cdot 10^{105\,585\,041}$
1 000	15.453	$1.6144 \cdot 10^{204}$	527 875 035	15.108	$1.1298 \cdot 10^{105\,585\,042}$
10 000	15.150	$2.3460 \cdot 10^{2\,005}$	1 000 000 000	15.108	$5.0463 \cdot 10^{200\,019\,001}$

The dependence of $K_1(m)$ on X_0 is negligible.

For each m we now have proved Theorem 3(a), that there are only finitely many m-cycles. Moreover, we can derive explicit upper and lower bounds for K, L and $x_{\min}$, as stated in Theorem 3(d). For $m \le 515\,619$ we can obtain better results in the next section.

Proof of Theorem 3(d) for $m \ge 515\,620$. The upper bound for K follows from Lemma 14, and the observation that k_1 is a decreasing function of m. The lower bound for K follows by Corollary 11. The bounds for L and $x_{\min}$ follow by combining this with Lemma 8 and Corollary 13. □

Moreover, we can combine Lemma 14 with the generalized Crandall Lemma 10, to prove that for small m there are no solutions at all. The next lemma proves a part of Theorem 3(b).

Lemma 15
There are no nontrivial m-cycles for $2 \le m \le 63$.

Proof. By Lemma 14, for $m \le 63$ we have $K < K_1(m) \le K_1(63) < 4.8181 \cdot 10^{15}$. This contradicts the $K > 5.7509 \cdot 10^{15}$ from Corollary 11 to the generalized Crandall Lemma 10. □

Note that Corollary 2 to Crandall's original Lemma 1 gives a result only for $m \le 30$.

The maximum value of m in Lemma 15 does depend heavily on the value of X_0, in the sense that any substantial improvement of the value of X_0 immediately leads to an improvement of the upper bound for m for which the proof works.

6.2 Reduced upper bound

Next we use, like Steiner [St], a continued fraction argument to find a better upper bound for K. We computed the continued fraction of δ up to $a_{200\,001}$. Using Mathematica 7.0 on a 2.1GHz Intel Core2 Duo T7500 processor this computation took less than 5 seconds.

Let $x = J_2(m)$ be the largest solution of

$$mc_m 2^{-\frac{\delta-1}{\delta^m-1}x} = \frac{\log 2}{2x}.$$

Lemmas 7 and 9(a) imply that $\dfrac{K+L}{K}$ is a convergent to δ whenever $K > J_2(m)$. Note that only convergents with odd index are of interest, as the sign of $p_n - q_n\delta$ alternates (Lemma 9(d)), and we have $\Lambda > 0$.

In view of Lemma 9(b), of particular interest are partial quotients a_n that are champions in the sense that $a_k < a_n$ whenever $k < n$, for even k, n.

For each m let $n(m)$ be the index n of the smallest champion for which $K_1(m) < q_{n-1}$. Then we define $A(m) = \max\{a_0, a_2, \ldots, a_{n(m)-2}\}$. Clearly $A(m)$ is the champion before $n(m)$. Indeed, we have

	$n(m)$	$A(m)$	$K_1(m) <$	$q_{n(m)-1} >$
$64 \le m \le 511$	218	55	$1.3035 \cdot 10^{106}$	$1.3133 \cdot 10^{106}$
$512 \le m \le 551$	230	100	$1.4043 \cdot 10^{114}$	$1.7807 \cdot 10^{114}$
$552 \le m \le 816$	330	964	$2.0804 \cdot 10^{167}$	$2.6341 \cdot 10^{167}$
$817 \le m \le 1\,340$	528	2 436	$2.1843 \cdot 10^{272}$	$2.3119 \cdot 10^{272}$
$1\,341 \le m \le 7\,009$	2 764	3 308	$9.1127 \cdot 10^{1\,406}$	$1.2197 \cdot 10^{1\,407}$
$7\,010 \le m \le 11\,143$	4 312	4 878	$1.0938 \cdot 10^{2\,234}$	$1.4973 \cdot 10^{2\,234}$
$11\,144 \le m \le 54\,234$	21 150	8 228	$5.5425 \cdot 10^{10\,853}$	$8.2400 \cdot 10^{10\,853}$
$54\,235 \le m \le 315\,502$	122 416	59 599	$1.1752 \cdot 10^{63\,113}$	$1.2759 \cdot 10^{63\,113}$
$315\,503 \le m \le 515\,619$	200 002	104 733	$3.0475 \cdot 10^{103\,140}$	$3.5522 \cdot 10^{103\,140}$

For $m \ge 515\,620$ we can in principle find corresponding values for $A(m)$ when we compute more partial quotients.

With the part of the continued fraction that we have at our disposal, we can derive a sharper upper bound for all $m \le 515\,619$, as follows. Let $x = K_2(m)$ be the largest solution of

$$mc_m 2^{-\frac{\delta-1}{\delta^m-1}x} = \frac{\log 2}{(A(m)+2)x}.$$

Note that $j_2(m) = J_2(m)/(m\delta^m)$ is a bounded function, that tends to $\log\delta/((\delta-1)\log 2) = 1.1358\ldots$ when $m \to \infty$. Also put $k_2(m) = K_2(m)/(m\delta^m)$. As we don't know how $A(m)$ will

grow when m increases, we don't know the exact behaviour of $k_2(m)$, but most probably it will also tend to $\log\delta/((\delta-1)\log 2) = 1.1358\ldots$ when $m \to \infty$. Indeed:

m	$j_2(m) <$	$J_2(m) <$	$k_2(m) <$	$K_2(m) <$
64	1.4664	$5.9379 \cdot 10^{14}$	1.5988	$6.4742 \cdot 10^{14}$
67	1.4547	$2.4553 \cdot 10^{15}$	1.5810	$2.6686 \cdot 10^{15}$
68	1.4510	$3.9396 \cdot 10^{15}$	1.5754	$4.2776 \cdot 10^{15}$
90	1.3881	$1.2542 \cdot 10^{20}$	1.4817	$1.3388 \cdot 10^{20}$
91	1.3859	$2.0068 \cdot 10^{20}$	1.4784	$2.1408 \cdot 10^{20}$
100	1.3677	$1.3737 \cdot 10^{22}$	1.4518	$1.4582 \cdot 10^{22}$
511	1.1964	$9.9079 \cdot 10^{104}$	1.2126	$1.0043 \cdot 10^{105}$
512	1.1963	$1.5734 \cdot 10^{105}$	1.2153	$1.5984 \cdot 10^{105}$
551	1.1927	$1.0669 \cdot 10^{113}$	1.2103	$1.0827 \cdot 10^{113}$
552	1.1926	$1.6940 \cdot 10^{113}$	1.2203	$1.7333 \cdot 10^{113}$
816	1.1766	$1.5769 \cdot 10^{166}$	1.1953	$1.6020 \cdot 10^{166}$
817	1.1765	$2.5023 \cdot 10^{166}$	1.1980	$2.5480 \cdot 10^{166}$
1 000	1.1701	$1.2224 \cdot 10^{203}$	1.1876	$1.2407 \cdot 10^{203}$
1 340	1.1625	$1.6517 \cdot 10^{271}$	1.1756	$1.6703 \cdot 10^{271}$
1 341	1.1625	$2.6198 \cdot 10^{271}$	1.1761	$2.6506 \cdot 10^{271}$
7 009	1.1422	$6.8625 \cdot 10^{1\,405}$	1.1448	$6.8782 \cdot 10^{1\,405}$
7 010	1.1422	$1.0879 \cdot 10^{1\,406}$	1.1449	$1.0905 \cdot 10^{1\,406}$
10 000	1.1405	$1.7660 \cdot 10^{2\,004}$	1.1424	$1.7690 \cdot 10^{2\,004}$
11 143	1.1401	$8.2326 \cdot 10^{2\,232}$	1.1418	$8.2450 \cdot 10^{2\,232}$
11 144	1.1401	$1.3050 \cdot 10^{2\,233}$	1.1419	$1.3071 \cdot 10^{2\,233}$
54 234	1.1369	$4.1685 \cdot 10^{10\,852}$	1.1373	$4.1698 \cdot 10^{10\,852}$
54 235	1.1369	$6.6069 \cdot 10^{10\,852}$	1.1374	$6.6097 \cdot 10^{10\,852}$
100 000	1.1365	$9.0096 \cdot 10^{20\,006}$	1.1368	$9.0116 \cdot 10^{20\,006}$
315 502	1.1361	$8.8364 \cdot 10^{63\,111}$	1.1362	$8.8371 \cdot 10^{63\,111}$
315 503	1.1361	$1.4006 \cdot 10^{63\,112}$	1.1362	$1.4007 \cdot 10^{63\,112}$
515 619	1.1361	$2.2915 \cdot 10^{103\,139}$	1.1361	$2.2916 \cdot 10^{103\,139}$

The next lemma now gives a better upper bound for K in the range for which the continued fraction argument can be applied, given the part of the continued fraction we have computed.

Lemma 16
For $64 \le m \le 515\,619$ a possible m-cycle satisfies $K < K_2(m)$.

Proof. Note that $J_2(m) < K_2(m)$ for all m, so we may assume $K > J_2(m)$. Then by the definition of J_2, Lemma 7 implies $\Lambda < \dfrac{\log 2}{2K}$, and so Lemma 9(a) shows that $\dfrac{K+L}{K} = \dfrac{p_n}{q_n}$ for some odd n. From Lemma 14 we have $K < K_1(m) < q_{n(m)-1}$, implying $n \le n(m) - 2$. Hence $a_{n+1} \le A(m)$. The definition of K_2, Lemmas 9(b) and 7 then imply $K < K_2(m)$. □

A computation[3] shows that there are only 68 values of m between 64 and 515 619 for which there is a convergent $\dfrac{p_n}{q_n}$ with odd index and $J_2(m) < q_n < K_2(m)$. In all other cases $q_n > J_2(m)$ implies $q_n > K_2(m)$. As J_2 and K_2 are relatively close, we do not elaborate.

As a consequence of Lemma 16 we can now prove a further part of Theorem 3(d), giving improved upper bounds for K, L and $x_{\min}$ in the case of $91 \le m \le 515\,619$. For $m \le 90$ we wil achieve even better results in the next section.

Proof of Theorem 3(d) for $91 \le m \le 515\,619$. The upper bound for K follows from Lemma 16, and the observation that $k_2(m) < 1.4784$ (note that $k_2(m)$ is a decreasing function, except when $A(m)$ jumps to the next champion). The lower bound for K follows by Corollary 11. The bounds for L follow by combining this with Lemma 8.

[3] In fact, this part of our computations was by far the most time consuming, taking 92 minutes.

For $x_{\min}$ we now can find a rather sharp upper bound, as follows. Let n be the index such that $q_{n-1} < K \le q_n$. By $K < K_2(m) \le K_2(515\,619) < 2.2916 \cdot 10^{103\,139} < 7.5013 \cdot 10^{103\,139} < q_{199\,998}$ we have $n \le 199\,998$. From the continued fraction we find that for this range $(a_n+1)(a_{n+1}+2) \le 24\,298\,288$. Now Corollary 5 and Lemma 9(c) and (b) show that $x_{\min} < \dfrac{m}{\Lambda} < \dfrac{m}{(\log 2)|p_n - q_n\delta|} < \dfrac{m(a_{n+1}+2)q_n}{\log 2} < \dfrac{m(a_{n+1}+2)(a_n+1)q_{n-1}}{\log 2} < \dfrac{m(a_{n+1}+2)(a_n+1)K}{\log 2} < \dfrac{24\,298\,288}{\log 2} mK$, where we used $q_n = a_n q_{n-1} + q_{n-2} < (a_n+1)q_{n-1}$. The bound for $x_{\min}$ now follows at once. □

The improvement we reached here for the bound of $x_{\min}$ is substantial. Though the improvement of the bounds for K and L is to some extent marginal (we cannot improve on the dominating term δ^m), the improvement becomes significant for small m, when we combine Lemma 16 with the generalized Crandall Lemma 10, to get a result improving on Lemma 15. The next lemma proves another part of Theorem 3(b).

Lemma 17
There are no nontrivial m-cycles for $2 \le m \le 68$.

Proof. In view of Lemma 15 we assume $m \ge 64$. By Lemma 16, for $m \le 68$ we have $K < K_2(m) \le K_2(68) < 4.2776 \cdot 10^{15}$. This contradicts $K > 5.7509 \cdot 10^{15}$ from Corollary 11. □

The maximum value of m in Lemma 17 does depend on the value of X_0, in the sense that any substantial improvement of the value of X_0 leads to an improvement of the upper bound for m for which the proof works.

7 Elimination of small solutions

7.1 Convergents

The classic approach now is to show, based on Lemma 9(a) as well as on refinements for e.g. "secondary convergents", that possible solutions in certain ranges are (secondary) convergents, and then to check all such (secondary) convergents for Corollary 5. Instead we will use an approximation lattice method, that can be seen as a more powerful variant of the continued fraction method. See [dW, Section 1.4 and Chapter 3] for some background on approximation lattices.

7.2 Lattices

We look for solutions in an "approximation lattice", as follows.

For a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ we define the norm $\|\mathbf{x}\|$ as $\|\mathbf{x}\| = \max\{|x_1|, |x_2|\}$.

Put $C = \left\lfloor \dfrac{X_0}{m} \dfrac{\log 2}{2} K_2(m) \right\rfloor$. Now let

$$\Gamma = \begin{pmatrix} 1 & 0 \\ [C\delta] & C \end{pmatrix},$$

where $[\cdot]$ stands for rounding towards the nearest integer. Then we look at the lattice of the $\mathbb{Z}$-linear combinations of the columns of Γ. For a solution K, L satisfying Corollary 5, Lemma 7 and Lemma 16 we look at the lattice point

$$\mathbf{x} = \Gamma \begin{pmatrix} -K \\ K+L \end{pmatrix} = \begin{pmatrix} -K \\ \Lambda_0 \end{pmatrix},$$

where $\Lambda_0 = (K+L)C - K[C\delta]$ is an approximation of $\dfrac{C}{\log 2}\Lambda$. Indeed, by Lemma 16 we have

$$\left|\Lambda_0 - \frac{C}{\log 2}\Lambda\right| \le K\,|C\delta - [C\delta]| \le \frac{1}{2}K_2(m).$$

It follows by Corollary 5 and the definition of C that

$$|\Lambda_0| \le \frac{1}{2}K_2(m) + \frac{C}{\log 2}|\Lambda| < \frac{1}{2}K_2(m) + \frac{C}{\log 2}\frac{m}{X_0} \le K_2(m).$$

It follows that we only have to search for lattice points with norm at most $K_2(m)$.

To achieve this efficiently, we compute (by the Euclidean, i.e. continued fraction, algorithm) a reduced basis of the lattice. Let Γ_{red} be a matrix with as columns this reduced basis. The lattice point $\mathbf{x}$ can be expressed in the reduced basis as $\mathbf{x} = \Gamma_{\text{red}}\mathbf{z}$ for some $\mathbf{z} \in \mathbb{Z}^2$. Due to the reducedness of the lattice basis, the number of points to be searched is approximately $\dfrac{K_2(m)^2}{C} \approx \dfrac{m}{X_0}K_2(m)$. A brute force search for points $\mathbf{z} \in \mathbb{Z}^2$ such that $\|\Gamma_{\text{red}}\mathbf{z}\| \le K_2(m)$ is therefore efficient when $K_2(m)$ is not much larger than $\dfrac{X_0}{m}$. Each lattice point found can be checked for fulfilling Corollary 5 and Lemma 7.

Doing this search for $69 \le m \le 90$ we found the following results.

Lemma 18
(a) *With* $69 \le m \le 75$ *there are no nontrivial* m*-cycles.*
(b) *With* $76 \le m \le 77$ *the only possible nontrivial* m*-cycles satisfy*

m	K	L	*falls when* $X_0 \ge$	*year*
76, 77	117 972 833 293 231 014	69 009 683 580 368 485	$\{5.39, 5.46\} \cdot 2^{60}$	2011
76, 77	124 207 383 220 472 977	72 656 661 496 678 846	$\{9.39, 9.52\} \cdot 2^{60}$	2015
76, 77	130 441 933 147 714 940	76 303 639 412 989 207	$\{36.8, 37.3\} \cdot 2^{60}$	2040
77	254 649 316 368 187 917	148 960 300 909 668 053	$7.58 \cdot 2^{60}$	2013

(c) *With* $78 \le m \le 90$ *the only possible nontrivial* m*-cycles satisfy*

m	*# solutions*	$K \ge$	$K \le$	*falls when* $X_0 \ge$	*year*
78	6	111 738 283 365 989 051	397 560 349 370 386 783	$447 \cdot 2^{60}$	2409
79	9	111 738 283 365 989 051	658 444 215 665 816 663	$452 \cdot 2^{60}$	2414
80	12	111 738 283 365 989 051	1 049 770 015 108 961 483	$458 \cdot 2^{60}$	2419
81	19	111 738 283 365 989 051	1 453 564 914 406 590 229	$464 \cdot 2^{60}$	2424
82	33	111 738 283 365 989 051	2 515 804 029 370 035 638	$469 \cdot 2^{60}$	2429

m	*# solutions*	$K \ge$	$K \le$	m	*# solutions*	$K \ge$	$K \le$
83	52	$1.1173 \cdot 10^{17}$	$4.6403 \cdot 10^{18}$	87	352	$1.1173 \cdot 10^{17}$	$2.8240 \cdot 10^{19}$
84	86	$1.1173 \cdot 10^{17}$	$6.6281 \cdot 10^{18}$	88	565	$1.1173 \cdot 10^{17}$	$4.6006 \cdot 10^{19}$
85	136	$1.1173 \cdot 10^{17}$	$1.0604 \cdot 10^{19}$	89	909	$1.1173 \cdot 10^{17}$	$7.7693 \cdot 10^{19}$
86	220	$1.1173 \cdot 10^{17}$	$1.7102 \cdot 10^{19}$	90	1 456	$1.1173 \cdot 10^{17}$	$1.1442 \cdot 10^{20}$

For $76 \le m \le 77$ we give with each solution the minimal value for X_0 that has to be reached to show that the solution does not correspond to an m-cycle. We also give the year in which this is expected to happen when the current rate of checking per day $3.5 \cdot 10^{15}$ values for $x_{\min}$ is continued, according to [OS], resuming the computations on January 1, 2011.

For $78 \le m \le 90$ we only give for each m the number of solutions found, and the minimal and maximal K. For $m \le 82$ we also give the minimal value for X_0 that has to be reached to show that the solutions do not correspond to an m-cycle, and the year in which this is expected to happen. For $m \ge 83$ this year is always further away than 4400.

The entire computation for the proof of Lemma 18 took 3.2 seconds.

As a result we can now complete the proof of Theorem 3.

Proof of Theorem 3(b), (c) and (d) for $m \le 90$. (b) follows at once from Lemma 18(a). The values in (c) and upper bound in (d) for K follow from Lemma 18. The lower bound for K follows by Lemma 18. The bounds for L follow by combining this with Lemma 8. In (c) the upper bound for $x_{\min}$ is derived from Corollary 5.

In (d) the upper bound for $x_{\min}$ is derived as in the above proof of the other part of Theorem 3(d), as follows. Let n be the index such that $q_{n-1} < K \le q_n$. By $K < K_2(m) \le K_2(90) < 1.3388 \cdot 10^{20} < 2.0563 \cdot 10^{20} < q_{43}$ we have $n \le 43$. From the continued fraction we find that $(a_n+1)(a_{n+1}+2) \le 168$. Hence $x_{\min} < \frac{168}{\log 2} mK$. □

8 Conclusion

In Theorem 3, it is of interest to note that the border $m = 75/76$ between (b) and (c) is directly related to the size of the lower bound X_0 for $x_{\min}$ that comes from external brute force computations, see [OS] and [Ro]. We indicated above when this border will be crossed, assuming that these computations continue at the present speed, and no new ideas emerge. We expect, as indicated in Lemma 18, that $m = 76$ and $m = 77$ can be expected to be solved within 25 years, and that $m = 78$ and beyond will take a considerably larger effort to finish.

The border $m = 77/78$ between Theorem 3(c) and (d) depends only on the amount of space one is willing to spend on listing candidate solutions that cannot yet be ruled out.

The border $m = 90/91$ in Theorem 3(d) depends directly on X_0, but it depends also on the amount of computation one is willing to spend on finding small lattice points. As the difference in the upper bounds between (c) and (d) is in the constants only, we see sofar no need to do more extensive computations.

The border $m = 515\,619/515\,620$ in Theorem 3(d) is directly related to how far one wishes to compute the continued fraction expansion of δ. Note that the difference in the upper bounds is substantial.

The border $m = 343\,118\,772/343\,118\,773$ in Theorem 3(d) was introduced only because it has some impact on the lower bounds for K and L. It is therefore not of much significance.

It seems that with the present techniques one cannot go much further. To show the nonexistence of nontrivial m-cycles for essentially larger ranges of m an entirely new idea seems to be needed.

Note that if certain values for K and L can not be excluded by the approximation lattice method, they can in principle be analyzed for an integer solution of the matrix equation. Without any knowledge on k_i and ℓ_i such an analysis is very inefficient. Computational and heuristical evidence indicates that for given m and K the worst case values for k_i and ℓ_i can be estimated, and then for any m an efficient analysis of possible m-cycles may become feasible. We believe that Lemma 7 is almost sharp for the worst case, in the sense that at best we expect that the factor mc_m may be improved to a constant, but the exponential dependence on m in the term $2^{-\frac{\delta-1}{\delta^m-1}K}$ seems unavoidable. Given the state of affairs in transcendence theory, we think that further improvement to Theorem 3 should come from sharpening the lower bounds for K. We leave this for future research.

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